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A THEORETICAL STUDY OF LATERAL STABILITY WITH AN AUTOMATIC PILOT

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SUMMARY

The influence of automatic operation of the aileron and rudder controls on the lateral stability of an airplane is discussed. The control deflections are assumed to be proportional to the deviations and to the rates of deviation of the airplane from steady-flight conditions. The effects of changes in the types of deviation governing control application are considered.

For one simple method of control in which the aileron deflection is proportional to the angle of bank and the rudder deflection is proportional to the angle of yaw, the effect of lag in control application is studied and regions of stability with and without lag are given. For the simple control with lag, curves are included that show the variations in the roots of the stability equation with changes in the amount of control applied.

It is concluded that, although the simple control provides a satisfactory means of varying most of the lateral-stability characteristics, the stability in azimuth will always be poor for such a control. Modification of the simple control by deflecting the ailerons in proportion to the angle of yaw appears to offer a promising method of improving the azimuth stability.

INTRODUCTION

The automatic control of aircraft has long been of interest and, as a result, considerable literature exists on the subject. Much of the published work, however, is confined to descriptions of devices proposed or employed to overcome the mechanical problems encountered in various specific types of automatic pilot. Theoretical treatments of the application of automatic control to aircraft have mainly consisted of general discussions of the differential equations of motion for the controlled airplane. Relatively few writers have presented data to show the influence of definite types of automatic control on the characteristics of specific airplanes. Such an investigation for longitudinal motion was made at New York University and the results are presented in reference 1.

In the present study, the influence of various methods of automatic control on the lateral stability of an airplane has been analytically determined. For the purposes of the investigation, a hypothetical airplane of average physical form was considered. The theoretical treatment employed is based primarily on well-known methods and assumptions used in studies of lateral stability. An outline of the general method of theoretical treatment used is given in the appendix together with definitions of the symbols involved.

AIRPLANE USED AS BASIS OF CALCULATIONS

Physical characteristics.—The airplane chosen as a basis of computations is the hypothetical average airplane discussed in reference 2. The characteristics, including the control characteristics, are similar to those of the Fairchild 22. As the properties assigned to the airplane are based on the average for many conventional airplanes, it should be possible to apply the general conclusions reached to all conventional designs.

The forces and the moments acting on a given airplane as a result of known linear and angular velocities of the airplane relative to the air can be determined from the nondimensional stability derivatives of the airplane, which are fixed by its physical form. For the study of automatic control, it has been found convenient to make use of similar nondimensional control derivatives, expressing the forces and the moments acting on the airplane as a result of control deflections. Values of the stability and the control derivatives for the average airplane are given in tables I and II for various flight conditions; these values are useful in comparing the characteristics of the airplane treated in this investigation with those of other airplane designs. The value 0.35 for the lift coefficient C_L is assumed to represent the condition of cruising flight; 1.0, the gliding condition; and 1.8, low-speed flight with flaps down. Table II also contains values of the velocity V along the flight path and values of the time-conversion factor $\tau (= m/\rho S V)$.

TABLE I

STABILITY DERIVATIVES FOR THE AVERAGE AIRPLANE

C_L	y_v	l_v	l_p	l_r	n_v	n_p	n_r
0.35	−0.140	−1.42	−4.43	0.905	0.960	−0.169	−0.744
1.0	−.200	−1.83	−4.46	2.60	1.02	−.416	−.916
1.8	−.415	−2.73	−4.56	4.65	1.31	−.574	−1.81

TABLE II

CONTROL DERIVATIVES AND VALUES OF V AND τ FOR THE AVERAGE AIRPLANE

C_L	y_{δ_r}	l_{δ_a}	n_{δ_a}	n_{δ_r}	V (fps)	τ (sec)
0.35	−0.0347	2.10	−0.106	0.474	150	0.815
1.0	−.0347	2.09	−.300	.475	88.5	1.38
1.8	−.0347	2.11	−.298	.473	66.0	1.85

In addition to the data contained in tables I and II, the following physical characteristics were assumed for the average airplane:

Wing span, b_____feet__ 32.0
Wing area, S_____square feet__ 171
Wing loading, W/S_____pounds per square foot__ 9.36
Ratio of wing span to radius of gyration about X, b/k_X__ 6.47
Ratio of wing span to radius of gyration about Z, b/k_Z__ 5.47
Relative-density factor, $\mu(=m/\rho Sb)$_____ 3.82

Inherent-stability characteristics.—In studies of stability, only the inherent stability of an airplane is usually determined, that is, the stability with the control surfaces fixed in their neutral positions. When the question of automatic control is considered, the matter of inherent stability is still of interest inasmuch as the character of the motion with the controls fixed forms a useful basis for comparing the effectiveness of various control methods.

As is indicated in the appendix, the lateral-stability characteristics can be determined from the roots of a stability equation of the form

$$a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0 \qquad (1)$$

When the control surfaces are fixed, the coefficients a to f are functions only of the stability derivatives, the lift coefficient, and the density factor μ. (See equation (12) of the appendix.) The inherent-stability roots for the average airplane with controls fixed are given in table III.

TABLE III

INHERENT-STABILITY ROOTS FOR THE AVERAGE AIRPLANE

C_L	$\lambda_{1.2}$	λ_3	λ_4	λ_5
0.35	−0.409±1.99i	−4.49	−0.00661	0
1.0	−.541±2.31i	−4.57	.0736	0
1.8	−1.03 ±2.36i	−4.81	.0909	0

The pair of conjugate complex roots $\lambda_{1.2}$ in table III represents an oscillatory component of the motion usually called the lateral, or Dutch roll, oscillation. The airplane used in the calculations has characteristics generally considered satisfactory for this mode. The real root λ_3 shows the damping of rolling motion. The pilot is ordinarily unaware of this rolling component of the motion following a disturbance of the airplane because the mode is so highly damped. The root λ_4 indicates the degree of spiral stability present. At cruising speeds the average airplane has almost neutral spiral stability; and, at lower speeds, it becomes definitely

unstable spirally. For most airplanes, stability of this mode is either very poor or lacking. The root λ_5 defines the stability in azimuth, that is, the tendency to follow a given compass course. The uncontrolled airplane always has neutral stability in this mode, as indicated by the zero value for the root λ_5.

STABILITY WITH CONTROL

When automatic control is introduced, the $\lambda_{1.2}$ lateral oscillation is retained but the spiral-stability root λ_4 may be real or combined with λ_3 or with λ_5 to form either of two distinct types of oscillation, depending on the control assumptions. For control conditions such that the $\lambda_{3.4}$ or the $\lambda_{4.5}$ oscillation is present, it should be noted that the oscillation will exist in addition to the $\lambda_{1.2}$ oscillation. For some flight conditions, the $\lambda_{1.2}$ oscillation and the added $\lambda_{3.4}$ or $\lambda_{4.5}$ oscillation may have very similar characteristics as regards period and damping, so that distinguishing between them in flight becomes difficult.

The criterion used in judging the desirability of any of the methods of automatic control subsequently discussed was the extent to which it improved the stability characteristics of the average airplane. The factors governing control deflection in the types of control considered are given in table IV.

TABLE IV

DESCRIPTION OF AUTOMATIC CONTROLS

Type of control	Aileron deflection proportional to—	Rudder deflection proportional to—
Displacement and rate-of-displacement.	Displacement in bank; displacement in azimuth; sideslipping velocity; rolling velocity; yawing velocity.	Displacement in bank; displacement in azimuth; sideslipping velocity; rolling velocity; yawing velocity.
Cross-coupled_____	Displacement in bank; displacement in azimuth.	Displacement in bank; displacement in azimuth.
Simple_____	Displacement in bank_____	Displacement in azimuth.

Attention was chiefly confined to the simple control for which the effect on the stability characteristics was determined of varying the amount of control deflection resulting from a unit change in the quantity governing deflection. The effect of lag in control application was also investigated.

Questions of mechanical difficulty in obtaining the various methods of control have been given but little consideration. Only systems dependent for their operation on displacements or rates of displacement have been treated because past experience with automatic control has demonstrated that such displacements and rates of displacement can be detected with relatively simple mechanisms. (See reference 3.) Although the control deflections might also be made to depend on the accelerations in roll, yaw, and sideslip, no such methods of control were considered in the present study because preliminary investigation indicated that they would be of little assistance in improving the character of the motion of the airplane.

Displacement and rate-of-displacement control.— The most general type of control treated in the present investigation is the type for which both the aileron and the rudder deflections are varied for any deviation in the angle of bank or in the angle of yaw or for any change in the velocities in rolling, yawing, or side-slipping. (See table IV.) From a study of the characteristics of this type of control, it was found that the total damping of the motion of an airplane, indicated by the sum of the damping coefficients, can be increased over that provided by the inherent stability of the airplane only if the aileron deflection is a function of the rolling or the yawing velocities or if the rudder deflection is a function of the yawing or the sideslipping velocities. No detailed characteristics of this type of control were determined because the extensive calculations involved were considered to be unjustified owing to the impracticability of such a complicated control.

Cross-coupled control.— Study of the preceding type of control indicated that, by means of suitable control operation, the total damping of the motion can be increased over that available as a consequence of the inherent stability of an airplane. Because of the very large inherent damping in roll, however, it appeared desirable to investigate the possibility of using a relatively simple type of control that would distribute the inherent damping of the airplane more uniformly among all the components of motion.

The assumption that both the aileron and the rudder deflections were functions of both the angle of bank and the angle of yaw made it possible to solve the stability equation for the special case of equal real roots, thus assuring that the motions after a disturbance of the airplane would contain no oscillatory components and that all modes would be equally damped. This method of control is called the cross-coupled control. (See table IV.) Coupling to provide deflections of the same control surface for two different types of deviation of the airplane can be easily accomplished, as demonstrated by the Smith automatic control (reference 4) for which deflection of the rudder occurs for deviations in either the angle of yaw or the angle of bank.

The five equal roots of the stability equation had the value $\lambda = -1.06$ when the numerical data for the average airplane at $C_L = 0.35$ were used; and the control gearings, which express the amount of control deflection applied for a unit deviation of the airplane in bank or yaw, had the values:

$$\left. \begin{array}{l} \dfrac{\partial \delta_a}{\partial \phi} = -0.731 \\[2mm] \dfrac{\partial \delta_a}{\partial \psi} = -0.954 \\[2mm] \dfrac{\partial \delta_r}{\partial \phi} = -0.356 \\[2mm] \dfrac{\partial \delta_r}{\partial \psi} = 1.116 \end{array} \right\} \tag{2}$$

The control gearings that enabled the cross-coupled control to provide equal damping of all modes of the

motion at $C_L = 0.35$ caused instability when used at higher lift coefficients. It is felt that, as a minimum requirement, any fully satisfactory method of automatic contol should result in stable motion of the airplane at all flying speeds; further study of the cross-coupled control as a means of obtaining uniform distribution of damping was therefore abandoned and attention was directed to more simplified forms of control.

Simple control.— In a method of control that has been successfully used for the Sperry automatic pilot, the aileron deflection δ_a is proportional to the angle of bank and the rudder deflection δ_r is proportional to the angle of yaw. For such a control, which is called the simple control (see table IV), the reduced number of variables involved made feasible more extensive calculations than were made for the two previously discussed methods of control. The range of the control gearings $\partial \delta_a / \partial \phi$ and $\partial \delta_r / \partial \psi$ that would provide stability for all flight speeds was determined; and, subsequently, the stability characteristics were studied as the control gearings were varied within the stable range, instead of solving for values of the control gearings that would give certain preassigned stability characteristics to the airplane. It was impossible to solve for the case of equal roots because the conditions for equal roots lead to a greater number of equations than there are variables.

From reference 5, it can be shown that the conditions which must be met if the motion is to be stable are: a, b, d, and f shall be positive, and

$$\left. \begin{array}{l} (be - af) > 0 \\ (bc - ad)(de - cf) - (be - af)^2 > 0 \end{array} \right\} \tag{3}$$

where the quantities a to f are the coefficients of the stability equation (equation (1)). The coefficients of equation (1) are functions of the stability derivatives, the control derivatives, the lift coefficient, the density factor, and the control gearings. The form of the coefficients can be obtained from equations (16) of the appendix by assuming $\partial \delta_a / \partial \psi$ and $\partial \delta_r / \partial \phi$ in the equations to be zero. Thus, upon substitution of the numerical data for the average airplane, the coefficients, and hence the expressions of equation (3), can be converted to functions of the control gearings $\partial \delta_a / \partial \phi$ and $\partial \delta_r / \partial \psi$.

The vanishing of either of the expressions of equation (3) or of any of the coefficients a, b, d, or f of equation (1) indicates that some mode of the motion of the airplane will become neutrally stable. If the various expressions are set equal to zero and are solved for pairs of values of $\partial \delta_a / \partial \phi$ and $\partial \delta_r / \partial \psi$, boundaries for neutral stability can be defined. (See fig. 1.) Throughout the flight range, the axis $\partial \delta_r / \partial \psi = 0$ (from the condition $f = 0$) and the line representing

$$(bc - ad)(de - cf) - (be - af)^2 = 0$$

were found to be the only conditions influencing the region of stability; the other expressions define lines that lie outside the region thus bounded. Because

the boundaries of the stable region include most of the quadrant formed by the negative control-gearing axes, it appears that almost any negative values for the control gearings will lead to stable motion of the airplane. As is indicated in figure 1, the stable region is slightly more restricted at low speeds because the position of the second boundary varies with lift coefficient.

If pairs of values of the control gearings $\partial\delta_a/\partial\phi$ and $\partial\delta_r/\partial\psi$, represented by points lying within the stable region for $C_L=1.8$, are chosen, the motion of the airplane will be stable throughout the flight range. It is difficult, however, to judge the degree of stability at any given point within the region, especially as the region is unbounded on two sides. In order to furnish

FIGURE 1.—Region of stability for simple control operating without lag.

more detailed information on the character of the motion, the roots of the stability equation were determined at selected points in the region.

For typical installations of the simple type of automatic control, the control gearings will probably have values of the order of -1.0. Representative cross sections of the trends in stability characteristics were therefore obtained by studying the changes in the roots when the rudder gearing was held constant at -1.0 and the aileron gearing was varied from zero to progressively more negative values and, next, when the aileron gearing was fixed at -1.0 and the rudder gearing was made increasingly negative. From this study of the stability roots, the boundary of the stable region representing the condition

$$(bc-ad)\ (de-cf)-(be-af)^2=0$$

was found to be the locus of the values of the control gearings for which the $\lambda_{4.5}$ oscillation is neutrally stable. The axis $\partial\delta_r/\partial\psi=0$ defines conditions for which the λ_5 mode, that is, the motion in azimuth, will be neutrally stable. The study also indicated that, for the simple control, the aileron gearing is essential for stability only if the airplane is inherently spirally unstable.

The rudder gearing is required for stability because of the neutral stability in azimuth for the uncontrolled airplane.

For cruising conditions $(C_L=0.35)$, most of the important changes in the character of the motion were found to occur as the control gearings vary between zero and -1.0. As the trends in the stability roots, with the exception of the damping of the $\lambda_{1.2}$ and the $\lambda_{3.4}$ oscillations, are much the same as those discussed later for the simple control with lag, further discussion of the roots will be deferred until the next section of the paper.

Simple control with lag.—The operation of any actual automatic control involves lag of various sorts due to inertia of its component parts, lost motion, etc. With a human pilot, lag in control is also introduced because of the pilot's inability to respond instantly to stimulus. The lag characteristics in any given case will be dependent upon the nature of the phenomenon causing the lag. For example, the lag produced by the dead region (the region that must necessarily exist near the neutral position for the components of the automatic pilot in order that control will not be applied when the airplane is on course) will delay application of the control until the deviation of the aircraft has reached a certain value, regardless of the rate of deviation. On the other hand, lag due to inertia is independent of the amount of deviation.

The mathematical methods employed in treating the various types of lag lead, in many cases, to very laborious calculations. The type of lag, however, for which the control deflection at a given instant is assumed to be always proportional to the deviation existing a fixed time ϵ previous to the given instant can be readily treated if the time lag ϵ is of the order of 0.1 second or less. The use of such a type of fixed time lag furnishes a satisfactory approximation of the effect of other types of lag, such as that caused by inertia, if the lag is small compared with the short-period oscillations of the airplane.

For sinusoidal motions of the airplane, operation of the controls with a fixed time lag is equivalent to out-of-phase application of the controls. As the period of such motions becomes shorter, a small time lag becomes of increasing importance as a cause of instability because the control tends to aid rather than to resist the deviations of the airplane.

The coefficients of equation (1) for the case of lag were obtained as functions of the control gearings $\partial\delta_a/\partial\phi$ and $\partial\delta_r/\partial\psi$ in the same manner as for the simple control without lag. In order to simplify the calculations, it was assumed that the lag was the same for both the aileron and the rudder controls and had a value of 0.1 second. This value of the time lag was chosen as being undoubtedly the minimum amount that will be present with manual operation of the controls and as probably approximating the lag of usual installations for automatic control.

More precise treatment of lag is contingent upon more detailed knowledge of the response characteristics of types of automatic pilots now available if the lag assumptions made are to be representative of actual operating conditions. Inasmuch as little information appears to be available concerning the response characteristics of even the most widely used types of automatic pilots, the determination of the behavior of typical installations is recommended as part of the future study of automatic control.

For the simple control with lag, the total damping of the motion depends on the magnitude of the control gearings and the amount of lag, in contrast to the case of the simple control without lag where no change in the total damping of the airplane is involved. The signs and the magnitudes of the terms involved are such that any lag will always reduce the total damping present. (See equation (23) of the appendix.)

The criterions for stability are the same as for the simple control without lag. By means of these criterions, the boundaries for neutral stability were obtained as indicated in figure 2 by a method analogous to that used for the simple control with no lag.

Comparison of the stable region shown in figure 2 with the stable region for the same control without lag (see fig. 1) shows that the introduction of lag limits the range of negative values of the control gearings for which the motion will be stable. The two boundaries present for the case of no lag also appear for the case with lag; two additional boundaries, however, appear when lag is involved, resulting in a completely closed stable region. Except for a portion of the boundary representing the condition

$$(bc-ad)(de-cf)-(be-af)^2=0$$

the stable region is a minimum for high-speed flight conditions. In order to define the region for which the motion will be stable throughout the range of lift coefficients investigated, part of the boundary for the condition $(bc-ad)(de-cf)-(be-af)^2=0$ at $C_L=1.8$ was included in figure 2.

The types of instability existing at the various boundaries of the stable region in figure 2 were determined by a study of the roots of the stability equation for control gearings represented by points on the boundaries. The boundary in figure 2 that is nearly parallel to the axis $\partial\delta_a/\partial\phi=0$ defines values of control gearings for which the $\lambda_{4\cdot5}$ oscillation will be neutrally stable and the boundary on the axis $\partial\delta_r/\partial\psi=0$ defines neutral stability in azimuth. These two boundaries are the ones that exist for no lag. Of the two remaining boundaries, the one that lies in the vicinity of the line $\partial\delta_a/\partial\phi=-4.7$ represents neutral stability of the $\lambda_{3\cdot4}$ oscillation and the other gives conditions for neutral stability of the $\lambda_{1\cdot2}$ oscillation.

For values of the control gearings lying within the stable region for $C_L=0.35$, the influence of systematic changes in control gearing on the roots of the stability equation was determined and is shown in figures 3 to 6, where the magnitudes of the real or the imaginary parts of the roots are given. A tendency toward instability for any mode is indicated by a decrease in magnitude of the real part, the mode becoming neutrally stable when the real part becomes zero. The period of the oscillatory modes grows longer as the magnitude of the imaginary part decreases, becoming infinite when the imaginary part is zero.

In order to facilitate conversion of the stability characteristics, expressed by the real and the imaginary parts of the roots, to the form involving the period and
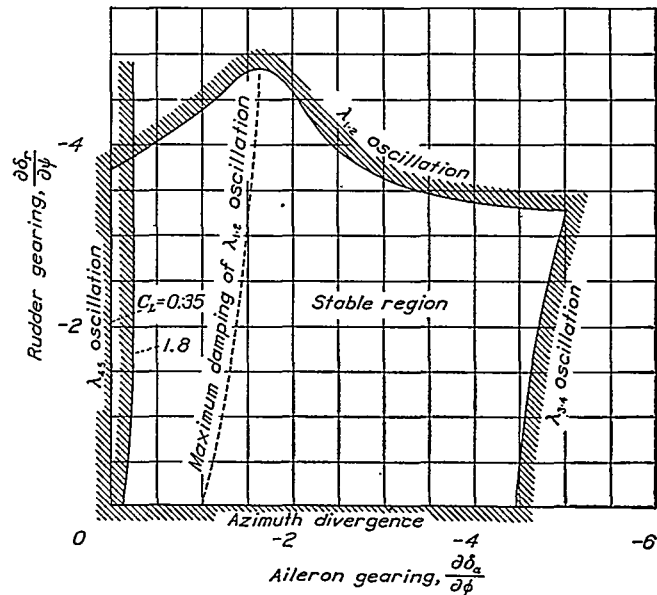

FIGURE 2.—Region of stability for simple control operating with 0.1-second lag.

the time to damp to one-half amplitude in seconds in cases where such a procedure might be desirable, the following data may be useful. For the airplane treated in the calculations, a value of 1.0 for the real part of the roots plotted in figures 3 to 6 is equivalent to a time to damp to one-half amplitude of 0.56 second; and a value of 1.0 for the imaginary part of the roots corresponds to an oscillation with a period of 5.1 seconds. The time in seconds varies inversely with the magnitude of the real and the imaginary parts of the roots for other values.

Figure 3 shows the effect of applying aileron control alone. When the aileron gearing is zero, the roots are those for the uncontrolled airplane. As the aileron gearing is made increasingly negative, the $\lambda_{1\cdot2}$ oscillation is relatively unaffected except that the damping of this mode is slightly improved for aileron gearings in the neighborhood of -1.0. The root λ_3, which is undesirably large for the uncontrolled airplane, rapidly decreases as small amounts of aileron gearing are introduced. The practically neutral inherent spiral stability of the uncontrolled airplane is replaced by considerable damping of this mode, as indicated by the changes in the root λ_4. When the aileron gearing is

about —0.6, the damping of the two aperiodic modes λ_3 and λ_4 becomes equal and they combine to form the $\lambda_{3\cdot4}$ oscillation. Further increase in the magnitude of the aileron gearing causes a rapid decrease in period and damping of the $\lambda_{3\cdot4}$ oscillation; this mode finally becomes unstable when the aileron gearing is about —4.5. The neutral stability in azimuth of the uncontrolled airplane is unaffected by the introduction of any amount of aileron control alone, as indicated by the absence of the λ_5 root in figure 3.

Figure 4 indicates the results of using rudder control alone. From the figure it is evident that the rudder control has a less complex effect on the character of the motion than the aileron control. The most important effect is the decrease in period and damping of the $\lambda_{1\cdot2}$ oscillation as the rudder gearing is made more negative; instability of this mode finally results when the rudder gearing is approximately —3.7. The damping-in-roll root λ_3 is unaffected by the value of the rudder gearing. When the rudder gearing is zero (representing the uncontrolled airplane), the root λ_4 is very small and the root λ_5 is zero. When the rudder gearing has any finite negative value, however, these two roots combine to form the $\lambda_{4\cdot5}$ oscillation, which has a very long period and practically zero damping. Thus the use of rudder control, when the aileron gearing is small or absent and when the inherent spiral stability is poor, will lead to a slow hunting of the airplane in its attempt to keep on a given compass course.

When the aileron and the rudder controls are used in combination, the separate influence of each control is still much the same as when the particular control is used alone. Comparison of figure 5 with figure 3 shows that the introduction of a fixed value of rudder gearing decreases the damping and the period of the $\lambda_{1\cdot2}$ mode for all values of the aileron gearing. The roots λ_4 and λ_5 are now also coupled to form an oscillation for small values of the aileron gearing and, at larger values of aileron gearing, the root λ_5 appears as a poorly damped subsidence. These effects may all be classified as influences of the added rudder control on the characteristics of the motion rather than as a modification of the influence of the aileron control due to the presence of the rudder, because the same modifications of the motions occur when rudder control alone is added to the uncontrolled airplane. Apparently the only influence that the rudder control has on the manner in which the aileron control affects the motions is that it somewhat increases the value of aileron gearing for which the damping of the $\lambda_{1\cdot2}$ oscillation is a maximum. In figure 2, combinations of aileron and rudder gearings for maximum damping of this mode are indicated by a dashed line.

Figure 6 shows the result of varying the rudder gearing in conjunction with a fixed value of aileron gearing. The influence of the rudder control on the $\lambda_{1\cdot2}$ oscillation is the same as it was with no aileron control present, the oscillation finally becoming unstable owing to the re-

duction in damping accompanying any increase in rudder gearing. The aileron gearing assumed $\left(\dfrac{\partial \delta_a}{\partial \phi}=-1.0\right)$ causes the roots λ_3 and λ_4 to couple as a well-damped oscillation with a period of approximately 2.47 seconds. Except for a slight increase in damping, this mode is practically unaffected by an increase in the rudder gearing. Because of the coupling of the λ_3 and the λ_4 roots brought about by the aileron control, the λ_5 root now appears as a separate subsidence instead of being coupled in the $\lambda_{4\cdot5}$ oscillation, as was the case for rudder control alone. The damping of the aperiodic component of motion represented by λ_5 (which mainly affects the azimuth stability) slowly increases as the rudder gearing is made more negative but never becomes of satisfactory magnitude.

Comparison of the changes in the roots of the stability equation brought about by variations in the control gearings used for the simple control indicates that the influence of the controls on the character of the motion is, in a general way, the same whether or not lag in the operation of the controls is assumed. The $\lambda_{1\cdot2}$ oscillation is always present and is mainly affected by the rudder control. As the rudder gearing is made more negative, the period of the oscillation steadily decreases; if there is no lag, the damping only slightly decreases. With the introduction of lag, however, instability of this mode will occur as the rudder gearing is increased in magnitude. The onset of instability is hastened as a result of the shortening of the period of the $\lambda_{1\cdot2}$ oscillation with increase in the rudder gearing. For short-period oscillations, a small amount of lag will result in serious out-of-phase application of the control. Even if the period of the $\lambda_{1\cdot2}$ oscillation did not decrease so that the phase shift remained small, the energy supplied to the oscillation still would be sufficient to cause instability for rudder gearings of large magnitude.

The coupling of the remaining roots λ_3, λ_4, and λ_5 of the stability equation depends on the magnitude of the aileron and the rudder gearings used. For small gearings (below about —1.0), the roots are chiefly dependent on the value of the aileron gearing and are apparently little affected by small amounts of lag, such as the lag assumed in the calculations. For aileron gearings more negative than about —0.6, the roots λ_3 and λ_4 are coupled as the $\lambda_{3\cdot4}$ oscillation and the λ_5 root appears as an aperiodic mode. The character of the $\lambda_{3\cdot4}$ oscillation depends almost entirely on the value of the aileron gearing. Increase in the magnitude of the aileron gearing causes a rapid decrease in the period of the $\lambda_{3\cdot4}$ oscillation and, if there is no lag, a very slight increase in damping. If lag is present, however, the damping of this mode decreases with increase in aileron gearing and the oscillation finally becomes unstable for large aileron gearings in the same manner that large rudder gearings cause instability of the $\lambda_{1\cdot2}$ oscillation when lag is present.
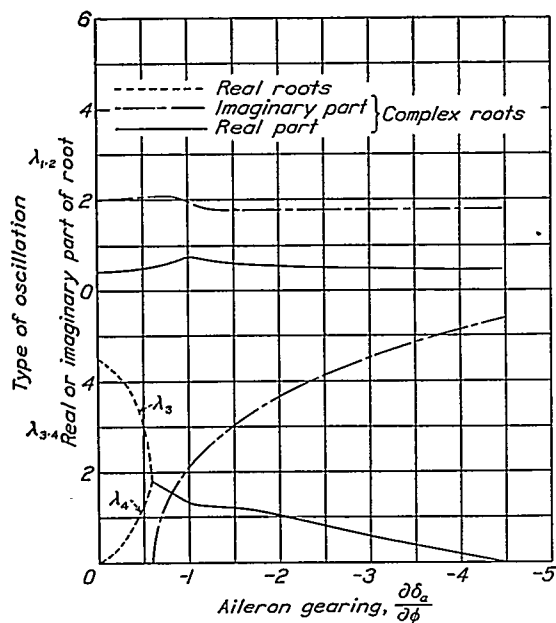
FIGURE 3.—Variation of roots with aileron gearing for simple control operating with 0.1-second lag. $\frac{\partial \delta_r}{\partial \psi}=0$.
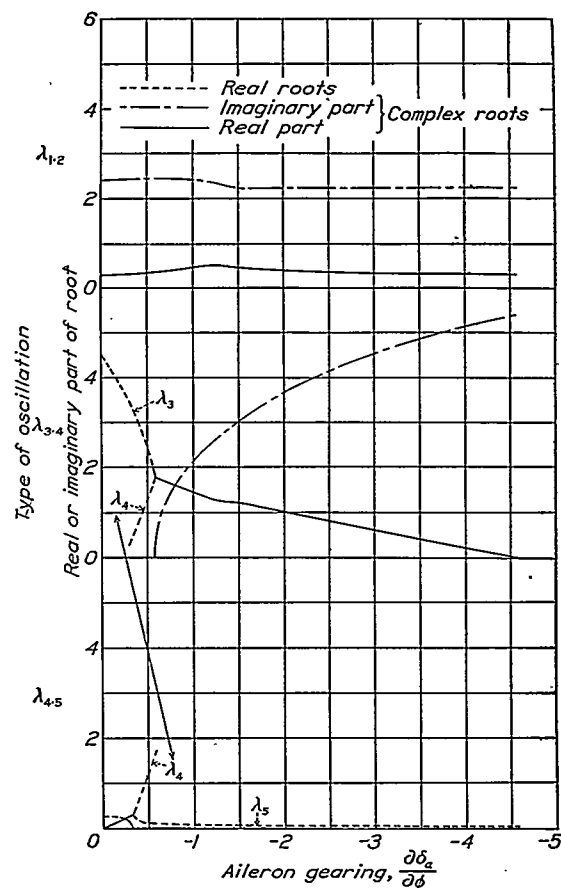

FIGURE 5.—Variation of roots with aileron gearing for simple control operating with 0.1-second lag. $\frac{\partial \delta_r}{\partial \psi}=-1.0$.
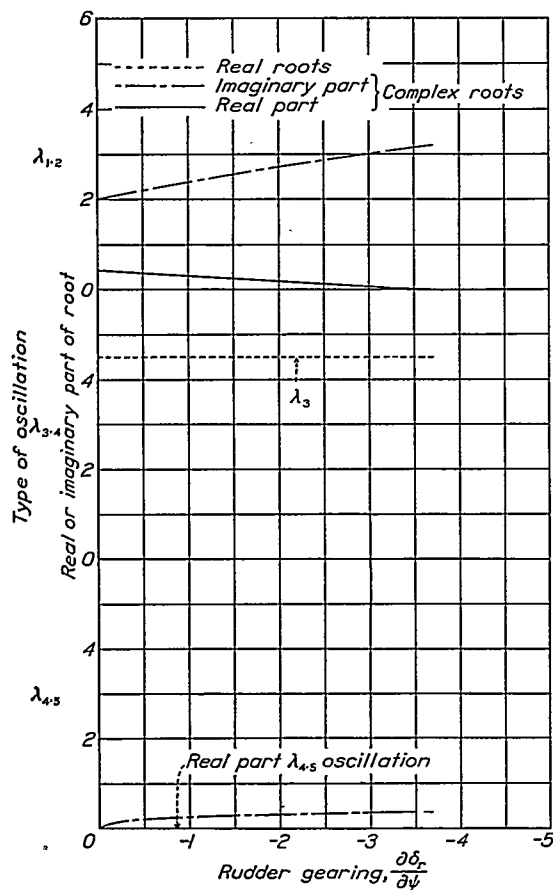

FIGURE 4.—Variation of roots with rudder gearing for simple control operating with 0.1-second lag. $\frac{\partial \delta_a}{\partial \phi}=0$.
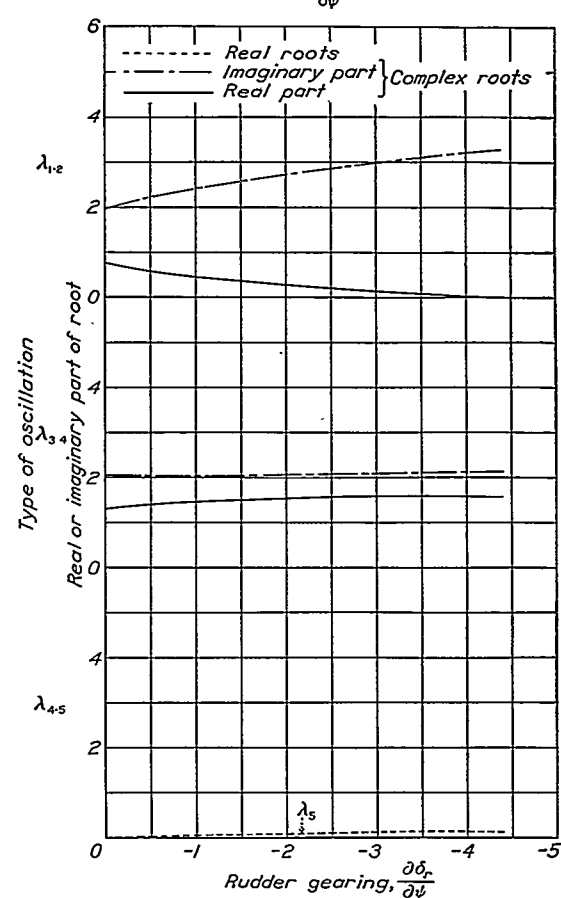

FIGURE 6.—Variation of roots with rudder gearing for simple control operatin gwith 0.1-second lag. $\frac{\partial \delta_a}{\partial \phi}=-1.0$.

For combinations of aileron and rudder gearings for which λ_5 exists as a real root, the damping of the mode slowly decreases with increase in aileron gearing at about the same rate that the damping increases with increase in rudder gearing. The variation of the root with changes in the control gearings appears to be unaffected by lag.

Improvement of the stability in azimuth.—Study of the simple control indicates that, although most of the characteristics of the motion of an airplane can be varied at will with this method of control by proper choice of values for the control gearings, the stability in azimuth will be poor for all usable control-gearing values, as is shown by the poor damping of the λ_5 or $\lambda_{4.5}$ mode. The magnitude of the constant term in the stability equation gives an indication of the stability in azimuth. From the evaluation of this term for the displacement and the rate of displacement control (see equations (14) of the appendix), the stability in azimuth can apparently be increased over its value for the simple control only if the ailerons are operated in proportion to the angle of yaw or sideslip or if the rudder is operated in proportion to the angle of bank or sideslip in conjunction with ailerons operated in proportion to the angle of yaw.

From a consideration of the magnitude of the terms involved, operating the ailerons in proportion to the angle of yaw was concluded to be the most promising method of improving the poor stability in azimuth that results from use of the simple control. As a preliminary study of the effectiveness of this method of improving the stability in azimuth, the roots of the stability equation were determined on the assumption that the simple control was augmented by the gearing $\partial\delta_a/\partial\psi$. The value -1.0 was assigned to all three control gearings. The stability and the control derivatives used were those for the average airplane at $C_L=0.35$, the azimuth stability having been found poorer at this lift coefficient than at $C_L=1.0$ or 1.8. The roots are given in table V, where they may be compared with the roots for a simple control for which $\partial\delta_a/\partial\phi$ and $\partial\delta_r/\partial\psi$ have the value -1.0 and the gearing $\partial\delta_a/\partial\psi$ is absent.

TABLE V

ROOTS OF THE STABILITY EQUATION FOR THE SIMPLE CONTROL WITH AND WITHOUT $\partial\delta_a/\partial\psi$ GEARING AT $C_L=0.35$. NO LAG IN CONTROL OPERATION

$\dfrac{\partial\delta_a}{\partial\psi}$	$\lambda_{1.2}$	$\lambda_{3.4}$	λ_5
-1.0	$-0.430\pm2.14i$	$-2.13\pm1.82i$	-0.202
0	$-.541\pm2.42i$	$-2.08\pm1.53i$	$-.069$

Comparison of the roots for the two methods of control shows a marked improvement of the stability in azimuth (increase in magnitude of λ_5) upon the addition of the $\partial\delta_a/\partial\psi$ gearing, although the characteristics of the other modes have not been materially altered.

The roots given in table V are for one random selection of values for the control gearings and, consequently, they are unlikely to represent the maximum improvement in the characteristics of the airplane motion that can be obtained by adding the $\partial\delta_a/\partial\psi$ gearing to the simple control. The values assumed for the control gearings in calculating the roots for the modified simple control were found to result in stable motion throughout the flight range.

A study of the factors involved indicates that the improvement of the damping in azimuth by adding the $\partial\delta_a/\partial\psi$ gearing to the simple control is contingent upon satisfactory inherent lateral weathercock stability of the airplane. If n_v has a reasonably large positive value (of the order of that possessed by the airplane considered in these calculations), the $\partial\delta_a/\partial\psi$ gearing can be very effective in improving the azimuth stability.

CONCLUSIONS

1. For the simple control, where the aileron deflection is proportional to the angle of bank and the rudder deflection is proportional to the angle of yaw, aileron gearing is essential for stability only if the airplane is inherently spirally unstable. The rudder control is required if the inherent neutral stability in azimuth of the airplane is to be remedied. The amount of aileron control used has a greater influence on the resulting character of the lateral motion than does the amount of rudder control used.

2. Although the simple control can provide stability throughout the flight range for all modes of the lateral motion of an airplane, relatively poor stability in azimuth will result from use of this type of control. Inasmuch as the other stability characteristics can be varied through a wide range by means of the simple control, a more complex method of control seems to be justified only if it improves the stability in azimuth.

3. Modification of the simple control, by provision for the operation of the aileron control in proportion to the angle of yaw, appears to offer a desirable method of improving the poor stability in azimuth resulting from the use of the simple control.

4. With the simple control, the presence of a small amount of lag should have a negligible effect if the control gearings are of normal magnitude. Lag becomes objectionable as the magnitudes of the control gearings are increased, the maximum values of the gearings that can be used without instability being governed by the amount of lag. Little improvement in the character of the motion is obtained, however, by the use of large control gearings.

APPENDIX

METHOD OF THEORETICAL TREATMENT

In theoretical studies of stability, it has been found convenient to make use of reference axes fixed relative to the structure of the airplane and so orientated that, in steady flight, the X, or longitudinal, axis is directed along the flight path and the Y axis is directed hori-
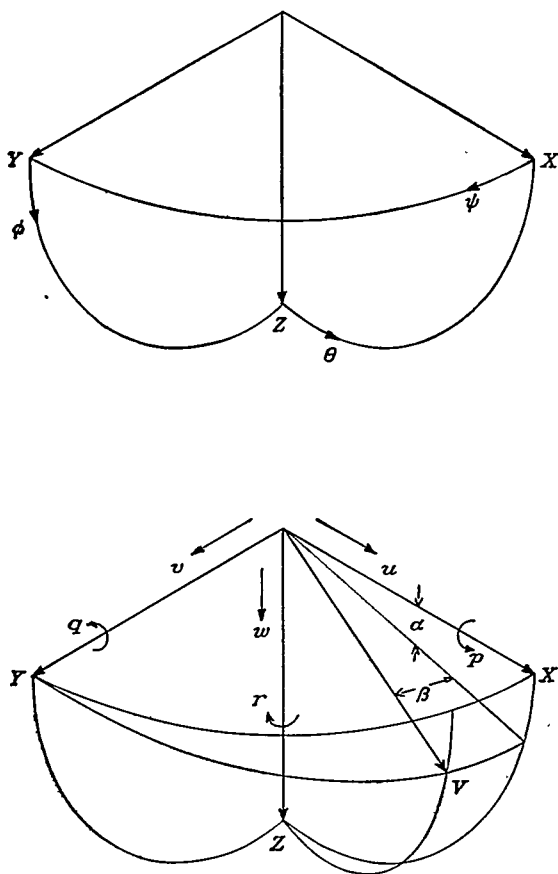

FIGURE 7.—Positive senses of axes and motions.

zontally along the span to the right. The Z axis is perpendicular to the X and the Y axes and directed downward in normal level flight so that the three axes form a right-hand system. Forces directed along the axes are also designated by the symbols X, Y, and Z and moments about the axes by L, M, and N, respectively. Changes in orientation of the reference axes from the initial position, due to small disturbances of

the airplane from steady flight, are defined by the angles θ, ϕ, and ψ in pitch, bank, and yaw, respectively. The initial orientation of the reference axes is assumed to be such that the initial angle of bank and angle of yaw are zero. The orientation of the reference axes relative to the instantaneous flight path is fixed by the angles α and β. The motion of the airplane is expressed by means of the linear velocity V of its center of gravity and the angular velocity Ω about its center of gravity. The axial components of V are u, v, and w and the axial components of Ω are p, q, and r. Positive senses of these various quantities are shown in figure 7. In addition to the preceding symbols, the equations of motion include: the mass m of the airplane; the wing span b; the wing area S; the radii of gyration k_X and k_Z about the X and Z axes, respectively; the acceleration due to gravity g; and the density of air ρ.

When the assumptions customarily employed in studies of lateral stability (reference 6) are used and it is assumed that, in addition to the aerodynamic and the inertia forces and moments acting because of the inherent-stability characteristics of the airplane, disturbing forces and moments which are functions of time [$Y(t)$, $L(t)$, and $N(t)$] are involved, the equations of lateral motion of an airplane can be written

$$mrV+m\frac{dv}{dt}=v\frac{\partial Y}{\partial v}+\phi mg+Y(t)$$
$$mk_x{}^2\frac{dp}{dt}=v\frac{\partial L}{\partial v}+p\frac{\partial L}{\partial p}+r\frac{\partial L}{\partial r}+L(t) \qquad (4)$$
$$mk_z{}^2\frac{dr}{dt}=v\frac{\partial N}{\partial v}+p\frac{\partial N}{\partial p}+r\frac{\partial N}{\partial r}+N(t)$$

The influence of the controls on the motion of an airplane can be studied by choosing suitable forms for the functions $Y(t)$, $L(t)$, and $N(t)$. Assume

$$Y(t)=\frac{\partial Y}{\partial\delta_a}\delta_a(t)+\frac{\partial Y}{\partial\delta_r}\delta_r(t)$$
$$L(t)=\frac{\partial L}{\partial\delta_a}\delta_a(t)+\frac{\partial L}{\partial\delta_r}\delta_r(t) \qquad (5)$$
$$N(t)=\frac{\partial N}{\partial\delta_a}\delta_a(t)+\frac{\partial N}{\partial\delta_r}\delta_r(t)$$

where $\delta_a(t)$ and $\delta_r(t)$ express the aileron and the rudder deflections, respectively, as functions of time. The

aileron deflection δ_a is positive when the right aileron is up and is the sum of the up and the down deflections of the two ailerons, expressed in radians. The rudder deflection δ_r, also expressed in radians, is positive when the trailing edge of the rudder is deflected to the right. Normally the side force due to aileron deflection and the rolling moment due to rudder deflection are small; equations (5) will therefore be simplified by neglecting the terms involving $\partial Y/\partial \delta_a$ and $\partial L/\partial \delta_r$. If the simplified forms of $Y(t)$, $L(t)$, and $N(t)$ in equations (5) are substituted in equations (4), the resulting equation for controlled lateral motion can be expressed in a simplified nondimensional form by the use of the equalities

$$p=\frac{d\phi}{dt}$$

$$r=\frac{d\psi}{dt}$$

and the substitutions

$$D=\frac{d}{dt}$$

$$\tau=\frac{m}{\rho SV}$$

$$\mu=\frac{m}{\rho Sb}$$

$$\beta=\frac{v}{V}$$

$$mg=\frac{1}{2}\rho V^2 S C_L$$

$$\frac{\partial Y}{\partial v}=\frac{1}{2}\rho VS\frac{\partial C_Y}{\partial \beta}$$

$$\frac{\partial L}{\partial v}=\frac{1}{2}\rho VSb\frac{\partial C_l}{\partial \beta}, \text{ etc.}$$

$$\frac{\partial L}{\partial p}=\frac{1}{4}\rho VSb^2\frac{\partial C_l}{\partial \frac{pb}{2V}}, \text{ etc.}$$

$$\frac{\partial Y}{\partial \delta_r}=\frac{1}{2}\rho V^2 S\frac{\partial C_Y}{\partial \delta_r}$$

$$\frac{\partial L}{\partial \delta_a}=\frac{1}{2}\rho V^2 Sb\frac{\partial C_l}{\partial \delta_a}, \text{ etc.}$$

where

$$C_L=\frac{\text{lift}}{\frac{1}{2}\rho V^2 S}$$

$$C_Y=\frac{Y}{\frac{1}{2}\rho V^2 S}$$

$$C_l=\frac{L}{\frac{1}{2}\rho V^2 Sb}$$

$$C_n=\frac{N}{\frac{1}{2}\rho V^2 Sb}$$

Equations (4) then become

$$
\left.
\begin{aligned}
&-\beta(y_v-\tau D)-\phi\left(\frac{C_L}{2}\right)-\psi(-\tau D)-y_{\delta_r}\delta_r(T)=0\\
&-\beta(\mu l_v)-\phi(l_p\tau D-\tau^2 D^2)-\psi(l_r\tau D)-\mu l_{\delta_a}\delta_a(T)=0\\
&-\beta(\mu n_v)-\phi(n_p\tau D)-\psi(n_r\tau D-\tau^2 D^2)-\\
&\qquad\qquad \mu[n_{\delta_a}\delta_a(T)+n_{\delta_r}\delta_r(T)]=0
\end{aligned}
\right\}
\tag{6}
$$

where

$$y_v=\frac{1}{2}\frac{\partial C_Y}{\partial \beta}$$

$$l_v=\frac{1}{2}\frac{b^2}{k_x^2}\frac{\partial C_l}{\partial \beta}$$

$$l_p=\frac{1}{4}\frac{b^2}{k_x^2}\frac{\partial C_l}{\partial \frac{pb}{2V}}$$

$$l_r=\frac{1}{4}\frac{b^2}{k_x^2}\frac{\partial C_l}{\partial \frac{rb}{2V}}$$

$$n_v=\frac{1}{2}\frac{b^2}{k_z^2}\frac{\partial C_n}{\partial \beta}$$

$$n_p=\frac{1}{4}\frac{b^2}{k_z^2}\frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$n_r=\frac{1}{4}\frac{b^2}{k_z^2}\frac{\partial C_n}{\partial \frac{rb}{2V}}$$

and

$$y_{\delta_r}=\frac{1}{2}\frac{\partial C_Y}{\partial \delta_r}$$

$$l_{\delta_a}=\frac{1}{2}\frac{b^2}{k_x^2}\frac{\partial C_l}{\partial \delta_a}$$

$$n_{\delta_a}=\frac{1}{2}\frac{b^2}{k_z^2}\frac{\partial C_n}{\partial \delta_a}$$

$$n_{\delta_r}=\frac{1}{2}\frac{b^2}{k_z^2}\frac{\partial C_n}{\partial \delta_r}$$

Factors y_v, l_v, etc., appearing in equations (6) are the nondimensional stability derivatives of the airplane. The nondimensional factors y_{δ_r}, l_{δ_a}, etc., expressing the forces and the moments acting on the airplane as a result of control deflections, will be called control derivatives. The time unit t has been replaced by the nondimensional time unit T, where $T=t/\tau$, so that $\tau D=\frac{d}{dT}$. The control deflections are now expressed as

functions of nondimensional time, using

$$\delta_a(t) = \underline{\delta}_a(T)$$
$$\delta_r(t) = \underline{\delta}_r(T)$$

In investigations of the influence of automatic control, the control-deflection functions $\underline{\delta}_a(T)$ and $\underline{\delta}_r(T)$ are conveniently defined in terms of certain of the deviations of the airplane from steady-flight conditions. If these definitions are used, equations (6) become a set of three simultaneous equations involving three independent variables: namely, the angle of sideslip β, the angle of bank ϕ, and the angular deviation in azimuth ψ.

The solutions of equations (6), expressing any one of the variables β, ϕ, or ψ as a function of time, are of the form

$$\beta, \phi, \text{ or } \psi = c_1 e^{\lambda_1 T} + c_2 e^{\lambda_2 T} + \ldots + c_5 e^{\lambda_5 T} + S(T) \quad (7)$$

where the λ's involved are the five roots of an equation obtained by writing the coefficients of equations (6) as a determinant, substituting $\lambda = \tau D$, and expanding to the form

$$a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0 \quad (8)$$

(See reference 7.)

In equation (7), the constants c_1 to c_5 merely determine the relative importance of any particular component of the motion in β, ϕ, or ψ. For a given airplane, the values of c_1 to c_5 are dependent on the particular degree of freedom in question, the method of control assumed, and the nature of the disturbance from steady-flight conditions. The term $S(T)$ in equation (7) is the so-called steady-state solution and represents conditions after the exponential components of the motion have died out. For the cases considered in this report, $S(T)$ is zero or a constant.

From the foregoing discussion, obviously only the roots of equation (8) are of interest in studying the stability of the motion, i. e., the tendency of the motion to decrease with time. Hence, equation (8) is frequently called the stability equation. The coefficients a to f of the stability equation are functions of the stability derivatives and of the factors involved in expressing the control-deflection functions $\delta_a(T)$ and $\delta_r(T)$ in terms of the deviations β, ϕ, and ψ of the airplane from steady flight. For a given airplane and a particular method of control operation, the coefficients, and hence the roots, of the stability equation are the same for all the solutions in β, ϕ, or ψ given by equation (7).

The roots of equation (8) can be either real or complex. Inasmuch as the coefficients a to f, however, are always real, complex roots can occur only as conjugate pairs and at least one root of the equation must always be either real or zero.

If A, B, and C are constants and if subscripts are used to identify the various roots of equation (8), for every pair of conjugate complex roots such as

$$\lambda_{1,2} = A \pm iB \quad (9)$$

it can be shown that the components of motion in equation (7) involving the complex pair of roots can be expressed as an oscillation such that

$$c_1 e^{\lambda_1 T} + c_2 e^{\lambda_2 T} = K e^{AT} \cos (BT + \xi) \quad (10)$$

where K and ξ are two new constants replacing c_1 and c_2. The amplitude of the oscillation defined by equation (10) diminishes one-half every $-0.693\tau/A$ second, and the period of the oscillation is $2\pi\tau/B$ seconds.

For every real root of equation (8) such as

$$\lambda_3 = C \quad (11)$$

an aperiodic component of the motion defined by equation (7) is present that loses one-half its amplitude every $-0.693\tau/C$ second.

Note that the expressions giving the time required to damp to one-half amplitude lead to negative values of time if A or C is positive. Hence, if any of the real roots or the real part of any of the complex roots of the stability equation is positive, the motion of the airplane will be unstable.

As previously mentioned, the method of applying control to the airplane depends on the form assumed for the functions $\delta_a(T)$ and $\delta_r(T)$ in equations (6). If these functions are assumed to be zero, equations (6) will define the motion of the airplane for the case where the controls are fixed in their neutral positions. The coefficients of equation (8) become, for fixed controls,

$$a_0 = 1$$
$$b_0 = -y_v - (l_p + n_r)$$
$$c_0 = (l_p n_r - l_r n_p) + y_v(l_p + n_r) + \mu n_v$$
$$d_0 = -y_v(l_p n_r - l_r n_p) + \mu(l_v n_p - l_p n_v) - \mu \frac{C_L}{2} l_v$$
$$e_0 = \mu \frac{C_L}{2}(l_v n_r - l_r n_v)$$
$$f_0 = 0$$

$$(12)$$

The subscripts given the coefficients are used to distinguish the values of the coefficients for the case of no control from the values to be given later for various methods of automatic control. Examination of equations (6) shows that no restoring force or moment proportional to ψ occurs unless it is furnished by a control operated so that its deflection is proportional to ψ; hence, in equation (8), the vanishing of the coefficient f_0 signifies that the airplane with fixed controls has neutral stability in azimuth.

A general form for the deflection functions is obtained by assuming that the control deflections are directly proportional to the displacements in bank and azimuth and also to the rolling, the yawing, and the sideslipping velocities of the airplane. The deflection

functions may then be expressed in nondimensional terms as

$$\underline{\delta}_a(T)=\beta\frac{\partial\delta_a}{\partial\beta}+\phi\frac{\partial\delta_a}{\partial\phi}+\frac{1}{2}\frac{1}{\mu}\tau D\phi\frac{\partial\delta_a}{\partial\frac{pb}{2V}}$$
$$+\psi\frac{\partial\delta_a}{\partial\psi}+\frac{1}{2}\frac{1}{\mu}\tau D\psi\frac{\partial\delta_a}{\partial\frac{rb}{2V}}$$
$$\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\quad(13)$$
$$\underline{\delta}_r(T)=\beta\frac{\partial\delta_r}{\partial\beta}+\phi\frac{\partial\delta_r}{\partial\phi}+\frac{1}{2}\frac{1}{\mu}\tau D\phi\frac{\partial\delta_r}{\partial\frac{pb}{2V}}$$
$$+\psi\frac{\partial\delta_r}{\partial\psi}+\frac{1}{2}\frac{1}{\mu}\tau D\psi\frac{\partial\delta_r}{\partial\frac{rb}{2V}}$$

The quantities $\partial\delta_a/\partial\beta$, $\partial\delta_a/\phi$, etc., expressing the amount of control deflection for a given deviation or rate of deviation of the airplane from steady-flight conditions, will be called control gearings.

If the deflection functions of equations (6) are assumed to be in the form given by equations (13), the coefficients of equation (8) become functions of the stability and the control derivatives and the control gearings. The expressions for the first two and the final coefficients are

$$a_1=a_0$$
$$b_1=b_0-\frac{1}{2}l_{\delta_a}\frac{\partial\delta_a}{\partial\frac{pb}{2V}}-\frac{1}{2}n_{\delta_a}\frac{\partial\delta_a}{\partial\frac{rb}{2V}}-y_{\delta_r}\frac{\partial\delta_r}{\partial\beta}-\frac{1}{2}n_{\delta_r}\frac{\partial\delta_r}{\partial\frac{rb}{2V}}$$
$$f_1=\mu^2\frac{C_L}{2}(l_v n_{\delta_a}-n_v l_{\delta_a})\frac{\partial\delta_a}{\partial\psi}$$
$$+\mu^2\frac{C_L}{2}l_v n_{\delta_r}\frac{\partial\delta_r}{\partial\psi}$$
$$-\mu^2[(l_v n_{\delta_a}-n_v l_{\delta_a})y_{\delta_r}+y_v l_{\delta_a}n_{\delta_r}]\left(\frac{\partial\delta_a}{\partial\phi}\frac{\partial\delta_r}{\partial\psi}-\frac{\partial\delta_a}{\partial\psi}\frac{\partial\delta_r}{\partial\phi}\right)$$
$$-\mu^2\frac{C_L}{2}l_{\delta_a}n_{\delta_r}\left(\frac{\partial\delta_a}{\partial\psi}\frac{\partial\delta_r}{\partial\beta}-\frac{\partial\delta_a}{\partial\beta}\frac{\partial\delta_r}{\partial\psi}\right)$$
$$\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}(14)$$

where a_0 and b_0 are the coefficients for no control given by equations (12). Expressions for the other coefficients of equation (8) are omitted because of their length.

From the theory of equations, for equation (8) b/a is known to equal the negative sum of all the roots of the equation or, in other words, to equal the total damping involved in any motion of the system represented by equations (6).

For the assumption that both the aileron-control and the rudder-control deflections are functions of both the angle of bank and the angle of yaw, the deflection functions have the form

$$\underline{\delta}_a(T)=\phi\frac{\partial\delta_a}{\partial\phi}+\psi\frac{\partial\delta_a}{\partial\psi}$$
$$\underline{\delta}_r(T)=\phi\frac{\partial\delta_r}{\partial\phi}+\psi\frac{\partial\delta_r}{\partial\psi}$$
$$\left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}\quad(15)$$

and the coefficients of equation (8) become

$$a_2=a_0$$
$$b_2=b_0$$
$$c_2=c_0-\mu l_{\delta_a}\frac{\partial\delta_a}{\partial\phi}-\mu n_{\delta_a}\frac{\partial\delta_a}{\partial\psi}-\mu n_{\delta_r}\frac{\partial\delta_r}{\partial\psi}$$
$$d_2=d_0+\mu[y_v l_{\delta_a}-(l_r n_{\delta_a}-n_r l_{\delta_a})]\frac{\partial\delta_a}{\partial\phi}$$
$$+\mu[y_v n_{\delta_a}+(l_p n_{\delta_a}-n_p l_{\delta_a})]\frac{\partial\delta_a}{\partial\psi}$$
$$-\mu(l_v y_{\delta_r}+l_r n_{\delta_r})\frac{\partial\delta_r}{\partial\phi}$$
$$+\mu[(y_v n_{\delta_r}-n_v y_{\delta_r})+l_p n_{\delta_r}]\frac{\partial\delta_r}{\partial\psi}$$
$$e_2=e_0+\mu[y_v(l_r n_{\delta_a}-n_r l_{\delta_a})+\mu(l_v n_{\delta_a}-n_v l_{\delta_a})]\frac{\partial\delta_a}{\partial\phi}$$
$$-\mu[y_v(l_p n_{\delta_a}-n_p l_{\delta_a})]\frac{\partial\delta_a}{\partial\psi}$$
$$+\mu[(l_v n_r-l_r n_v)y_{\delta_r}+(y_v l_r+\mu l_v)n_{\delta_r}]\frac{\partial\delta_r}{\partial\phi}$$
$$-\mu[(l_v n_p-l_p n_v)y_{\delta_r}+y_v l_p n_{\delta_r}]\frac{\partial\delta_r}{\partial\psi}$$
$$+\mu^2 l_{\delta_a}n_{\delta_r}\left(\frac{\partial\delta_a}{\partial\phi}\frac{\partial\delta_r}{\partial\psi}-\frac{\partial\delta_a}{\partial\psi}\frac{\partial\delta_r}{\partial\phi}\right)$$
$$f_2=f_0+\mu^2\frac{C_L}{2}(l_v n_{\delta_a}-n_v l_{\delta_a})\frac{\partial\delta_a}{\partial\psi}+\mu^2\frac{C_L}{2}l_v n_{\delta_r}\frac{\partial\delta_r}{\partial\psi}$$
$$-\mu^2[(l_v n_{\delta_a}-n_v l_{\delta_a})y_{\delta_r}+y_v l_{\delta_a}n_{\delta_r}]$$
$$\left(\frac{\partial\delta_a}{\partial\phi}\frac{\partial\delta_r}{\partial\psi}-\frac{\partial\delta_a}{\partial\psi}\frac{\partial\delta_r}{\partial\phi}\right)$$
$$\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\}\quad(16)$$

where a_0 to f_0 are the coefficients for no control given by equations (12).

If the five roots of equation (8) are equal and have the value λ, it is known from the theory of equations that the coefficients of equation (8) can be expressed in terms of λ as

$$\begin{aligned}a&=1\\ b&=-5\lambda\\ c&=10\lambda^2\\ d&=-10\lambda^3\\ e&=5\lambda^4\\ f&=-\lambda^5\end{aligned}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}\quad(17)$$

Thus, if the control deflections are assumed to have the form given by equations (15), five equations can be formed from equations (16) and (17) to determine the values of the four control gearings $\partial\delta_a/\partial\phi$, $\partial\delta_a/\partial\psi$, $\partial\delta_r/\partial\phi$, and $\partial\delta_r/\partial\psi$ and the value of λ.

TREATMENT OF LAG

The mathematical representation of lag in control application of the type in which the amount of control applied at a given instant is assumed to be proportional to the amount of deviation existing at a fixed time ϵ previous to the given instant can be readily represented by use of Taylor's expansion to express $f(t-\epsilon)$ in

terms of $f(t)$

$$f(t-\epsilon)=f(t)-\epsilon f'(t)+\ldots+\frac{(-\epsilon)^n}{n!}f^n(t)+\ldots \quad (18)$$

Using $D=\dfrac{d}{dt}$, $D^2=\dfrac{d^2}{dt^2}$, etc., equation (18) can be written

$$f(t-\epsilon)=e^{-\epsilon D}f(t) \quad (19)$$

The lag can be expressed in nondimensional time units by using $\lambda=\tau D$ thus:

$$f(t-\epsilon)=e^{-\frac{\epsilon}{\tau}\lambda}f(t)$$

or, if $f(t)$ is also expressed in nondimensional time units,

$$\underline{f}\left(T-\frac{\epsilon}{\tau}\right)=e^{-\frac{\epsilon}{\tau}\lambda}\underline{f}(T) \quad (20)$$

If a type of automatic control is assumed for which the aileron deflection is proportional to the angle of bank and the rudder control is proportional to the angle of yaw, the deflection functions can be obtained from equations (15) by assuming the control gearings $\partial\delta_a/\partial\psi$ and $\partial\delta_r/\partial\phi$ to be zero. When a lag of ϵ_1 seconds is introduced into the operation of the aileron control and a similar lag of ϵ_2 seconds is introduced into the operation of the rudder control, the deflection functions become

$$\left.\begin{array}{l}\delta_{\underline{a}}(T)=e^{-\frac{\epsilon_1}{\tau}\lambda}\,\phi\dfrac{\partial\delta_a}{\partial\phi}\\[2mm]\delta_{\underline{r}}(T)=e^{-\frac{\epsilon_2}{\tau}\lambda}\,\psi\dfrac{\partial\delta_r}{\partial\psi}\end{array}\right\} \quad (21)$$

For small amounts of lag, it was found that the exponentials in equations (21) could be evaluated with sufficient accuracy by using the first three terms of the power series expansions for the exponentials. The deflection functions then become

$$\left.\begin{array}{l}\delta_{\underline{a}}(T)=\dfrac{\partial\delta_a}{\partial\phi}\left[1-\dfrac{\epsilon_1}{\tau}\lambda+\dfrac{1}{2}\left(\dfrac{\epsilon_1}{\tau}\right)^2\lambda^2\right]\phi\\[3mm]\delta_{\underline{r}}(T)=\dfrac{\partial\delta_r}{\partial\psi}\left[1-\dfrac{\epsilon_2}{\tau}\lambda+\dfrac{1}{2}\left(\dfrac{\epsilon_2}{\tau}\right)^2\lambda^2\right]\psi\end{array}\right\} \quad (22)$$

For the deflection functions defined by equations (22), the expressions for the first two coefficients of equation (8) become (neglecting terms containing as a factor products of the lag beyond the second degree)

$$\left.\begin{array}{l}a_3=a_0-\dfrac{1}{2}\left(\dfrac{\epsilon_1}{\tau}\right)^2\mu l_{\delta_a}\dfrac{\partial\delta_a}{\partial\phi}-\dfrac{1}{2}\left(\dfrac{\epsilon_2}{\tau}\right)^2\mu n_{\delta_r}\dfrac{\partial\delta_r}{\partial\psi}\\[3mm]b_3=b_0+\dfrac{\epsilon_1}{\tau}\mu l_{\delta_a}\dfrac{\partial\delta_a}{\partial\phi}+\dfrac{\epsilon_2}{\tau}\mu n_{\delta_r}\dfrac{\partial\delta_r}{\partial\psi}\\[3mm]\qquad+\dfrac{1}{2}\left(\dfrac{\epsilon_1}{\tau}\right)^2\mu[y_v l_{\delta_a}-(l_r n_{\delta_a}-n_r l_{\delta_a})]\dfrac{\partial\delta_a}{\partial\phi}\\[3mm]\qquad+\dfrac{1}{2}\left(\dfrac{\epsilon_2}{\tau}\right)^2\mu[(y_v n_{\delta_r}-n_v y_{\delta_r})+l_p n_{\delta_r}]\dfrac{\partial\delta_r}{\partial\psi}\end{array}\right\} \quad (23)$$

where a_0 and b_0 are given by equations (12). The coefficients given in equations (23) are of particular interest because they are indicative of the total damping of the motion. Expressions for the other coefficients of equation (8) are omitted because of their length.

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